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On a variance associated with the distribution of general sequences in arithmetic progressions. II

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Asymptotic formulae of Montgomery–Hooley type are obtained for general sequences which, for relatively small moduli, have an approximate asymptotic distribution in each residue class.

Keywords: variance; distribution; sequences; residue classes; asymptotics

1. Introduction

We continue our study of the variance associated with the distribution of general sequences in arithmetic progresions. In particular, we turn our attention to the situation considered by Hooley (1975*a*). Thus, we are interested in the extent to which it is possible to obtain an asymptotic formula for the variance

$$V(x,Q)=\sum_{q\leqslant Q}\sum_{a\in\mathcal{A}(q)}|A(x;q,a)-f(q,a)\Phi(x)|^2, \tag{1.1}$$

where $\mathcal{A}(q)$ is a suitable set of residue classes modulo q, $A(x;q,a)$ denotes

$$A(x;q,a)=\sum_{\substack{n\leqslant x\\ n\equiv a\ (\mathrm{mod}\ q)}}a_n, \tag{1.2}$$

and f and Φ appropriately reflect the local and global properties, respectively, of the real sequence $\{a_n\}$.

As remarked upon in Vaughan (this volume), for reasons of practical expediency, it is usual to introduce a system of weights so that the main terms $f(q,a)\Phi(x)$ are transformed into $f(q,a)x$. Having illustrated this procedure at some length in our previous work, and there being no need to dwell on the point any further, we satisfy ourselves with the observation that such tranformations can also be applied in the work described herein and with the same general conclusions modulo any natural adjustments which may be required by the situation in hand.

The situation of greatest interest is normally that in which the set of residues $\mathcal{A}(q)$ includes all those a for which $A(x;q,a)$ has a positive asymptotic density as $x\to\infty$. Thus, where necessary by an appropriate adjustment to $f(q,a)$, we may suppose that $\mathcal{A}(q)$ is a complete set of residues modulo q. Finally, in the vast majority of cases of practical interest the 'local factor' $f(q,a)$ which arises depends on (a,q) rather than a (and when a_n is specialized to be the indicator function of a set of integers this is the precise situation which is studied by Hooley (1975*c*)). Thus in this memoir we

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suppose that $V(x,Q)$ satisfies not (1.1) but

$$V(x,Q)=\sum_{q\leqslant Q}\sum_{a=1}^{q}|A(x;q,a)-f(q,(q,a))x|^2. \tag{1.3}$$

Here we suppose that there is an increasing function $\Psi(x)$, with $\Psi(x)>\log x$ for all large x, $\Psi(1)>0$ and

$$\int_1^x \Psi(y)^{-1}\,\mathrm{d}y \ll x\Psi(x)^{-1},$$

such that

$$A(x;q,a)=xf(q,(q,a))+O\left(\frac{x}{\Psi(x)}\right) \tag{1.4}$$

uniformly for all real $x\geqslant 1$ and natural numbers q and a, and we note that immediately from these assumptions we have $\Psi(x)\ll x$.

The most natural assumption concerning the a_n in the argument we have in mind is not that the sequence a_n be the indicator function of a set but rather that it be bounded in mean square, or, more precisely, that

$$\sum_{n\leqslant x} a_n^2 \ll x \tag{1.5}$$

uniformly for all positive real x. The nature of our results depends on the properties of the arithmetical function

$$g(q)=\phi(q)\left(\sum_{r|q} f(q,r)\mu(q/r)\right)^2. \tag{1.6}$$

One consequence of (1.5) is that the series

$$\sum_{q=1}^{\infty} g(q)$$

converges, and the quality of our main conclusions depends on the rate of convergence of this series and the extent to which

$$x\sum_{q=1}^{\infty} g(q)$$

is a good approximation to the left-hand side of (1.5).

With the above definitions it is now possible to state a simple conclusion.

Theorem 1.1. *Suppose that (1.5) holds and that*

$$Q>\sqrt{x}\log 2x, \tag{1.7}$$

and let

$$E(z)=\int_1^z \sum_{q>y} g(q)\,\mathrm{d}y \tag{1.8}$$

and

$$U(x,Q)=V(x,Q)-Q\sum_{n\leqslant x} a_n^2 + Qx\sum_{q=1}^{\infty} g(q). \tag{1.9}$$

Then $E(z) = o(z)$ *as* $z \to \infty$ *and*

$$U(x,q) \ll x^{3/2}\log x + x^2(\log 2x)^{9/2}\Psi(x)^{-1} + x^2(\log x)^{4/3}\Psi(x)^{-2/3} + Q^2E(x/Q).$$

By comparison, Hooley (1975), in the special case that a_n is an indicator function and that (1.4) holds with $\Psi(x) = (\log x)^A$ for every fixed positive A, has shown that

$$V(x,Q) \ll Qx + x^2(\log x)^{-A}.$$

One might hope that the term containing $\Psi(x)^{-2/3}$ could be dispensed with, but all the internal evidence, either from the proof given here or Hooley's (1975) method, suggests that to be successful in this endeavour some information is required concerning the behaviour of the a_n in short intervals, i.e. of $A(u+v;q,a) - A(u;q,a)$.

By Parseval's identity

$$\int_0^1 |G(\alpha)|^2\,\mathrm{d}\alpha = \sum_{n \leqslant x} a_n^2,$$

where

$$G(\alpha) = \sum_{n \leqslant x} a_n e(n\alpha) \tag{1.10}$$

and it is not hard to show on assumption (1.4) that the contribution from the consequential natural major arcs is asymptotically

$$x\sum_{q=1}^{\infty} g(q).$$

Thus the main term

$$Q\sum_{n \leqslant x} a_n^2 - Qx\sum_{q=1}^{\infty} g(q)$$

in theorem 1.1 is closely related to the minor arcs. In many of the common situations matching our conditions it is known that the contribution from the minor arcs is smaller than that from the major arcs. For example, this is so when a_n is the indicator function of the k-free numbers ($k \geqslant 2$). Thus, in such a situation the two expressions in the main terms largely cancel. However, we can then anticipate that provided we have some knowledge of the asymptotic behaviour of their difference, and perhaps also of $E(y)$, it is still possible to obtain the asymptotic behaviour of $V(x,Q)$. That further information regarding $E(y)$ may be required is born out by the case of k-free numbers where the final main term is indeed of the same order of magnitude as $Q^2E(x/Q)$ for a large range of Q.

Theorem 1.2. *Suppose that there are positive real numbers* η *and* c *such that* $0 < \eta < 1$

$$\sum_{n \leqslant x} a_n^2 - x\sum_{q=1}^{\infty} g(q) = o(x^{(2+\eta)/(2+2\eta)}) \tag{1.11}$$

as $x \to \infty$ *and*

$$\sum_{q > y} g(q) \sim cy^{-\eta} \tag{1.12}$$

as $y \to \infty$. *Suppose further that*

$$Q > \sqrt{x} \log 2x.$$

Then

$$V(x, Q) = Q^2 M(x/Q) + O(x^{3/2} \log x + x^2 (\log 2x)^{9/2} \Psi(x)^{-1} + x^2 (\log x)^{4/3} \Psi(x)^{-2/3}),$$

where

$$M(y) \sim c \frac{-2\zeta(-\eta)}{1-\eta^2} y^{1-\eta}$$

as $y \to \infty$.

Theorem 1.2 can be applied in the case when a_n is the indicator function of the k-free numbers, and it gives a main term of the general form

$$CQ^2 (x/Q)^{1/k}.$$

However, it is possible to make use of the special features of the sequence of k-free numbers so as to obtain an error term superior to the one given here (see Croft 1975), and we intend to return to this problem in a future paper.

The method of proof of theorem 1.2 is equally valid under more general conditions than (1.11) and (1.12). For example, with appropriate adjustments to (1.11) and the conclusion, condition (1.12) could be replaced by

$$\sum_{q>y} g(q) \sim \kappa(y), \tag{1.13}$$

where $\kappa(y)$ is a suitably smooth function tending to zero through positive values.

It is natural to ask whether the main terms in theorem 1.1 always cancel, and we show that this is not so by the construction of an example. The point is that the example places a positive proportion of the mass in

$$\int_0^1 |G(\alpha)|^2 \, \mathrm{d}\alpha$$

on the minor arcs.

Theorem 1.3. *Let* $\lambda = \frac{1}{2}(\sqrt{5} - 1)$ *and* $\theta \in (0, 1)$, *and let* a_n *be* 1 *when* $\{\lambda n\} < \theta$ *and be* 0 *otherwise. Then (1.4) holds with* $f(q, (a, q)) = \theta/q$ *and* $\Psi(x) = x^{1/3}$, *and*

$$\sum_{q=1}^{\infty} g(q) = \theta^2,$$

but

$$\sum_{n \leqslant x} a_n^2 = \theta x + O(x^{2/3}).$$

We give an elementary proof of theorem 1.3. By invoking the Erdős–Turán theorem, or Selberg's magic functions, it would be possible to take $\Psi(x) = x^{1/2}$ and achieve an error $O(x^{1/2})$ in the final conclusion.

2. Preliminary lemmata

Here we record some useful consequences of assumptions (1.4) and (1.5).

Lemma 2.1. *Assume (1.4) and (1.5). Then*

$$\sum_{q\leqslant Q}\sum_{r|q}|f(q,r)|\frac{\sigma(q)}{q} \ll (\log 2Q)^{5/2} \tag{2.1}$$

and

$$\sum_{q\leqslant Q}\sum_{r|q}|f(q,r)|d(q/r) \ll (\log 2Q)^{7/2}. \tag{2.2}$$

Proof. We give first the proof of (2.2). When $r|q$ the number of integers a with $1\leqslant a\leqslant q$ and $(q,a)=r$ is $\phi(q/r)$. Thus

$$|f(q,r)| = \frac{1}{\phi(q/r)}\sum_{\substack{a=1\\(q,a)=r}}^{q}|f(q,(q,a))|.$$

Hence, by (1.4),

$$\sum_{q\leqslant Q}\sum_{r|q}|f(q,r)|d(q/r) = \lim_{x\to\infty}\lambda(x),$$

where

$$\begin{aligned}\lambda(x) &= x^{-1}\sum_{q\leqslant Q}\sum_{r|q}\frac{d(q/r)}{\phi(q/r)}\sum_{\substack{a=1\\(q,a)=r}}^{q}\left|\sum_{\substack{n\leqslant x\\ n\equiv a\ (\mathrm{mod}\ q)}}a_n\right|\\ &\leqslant x^{-1}\sum_{n\leqslant x}|a_n|\sum_{q\leqslant Q}\frac{d(q/(q,n))}{\phi(q/(q,n))}.\end{aligned}$$

The innermost sum here is

$$\sum_{\substack{r\leqslant Q\\ r|n}}\sum_{\substack{s\leqslant Q/r\\ (s,n/r)=1}}\frac{d(s)}{\phi(s)} \ll \sum_{\substack{r\leqslant Q\\ r|n}}(\log 2Q)^2.$$

Hence, by Cauchy's inequality and (1.5),

$$\lambda(x)^2 \ll x^{-1}\sum_{n\leqslant x}\Bigg(\sum_{\substack{r\leqslant Q\\ r|n}}1\Bigg)^2(\log 2Q)^4 \ll (\log 2Q)^7.$$

To prove (2.1) we argue in a similar vein. We observe that it suffices to bound

$$x^{-1}\sum_{n\leqslant x}|a_n|\sum_{q\leqslant Q}\frac{\sigma(q)}{q\phi(q/(q,n))}.$$

The inner sum here is

$$\ll \sum_{\substack{r\leqslant Q\\ r|n}}\frac{\sigma(r)}{r}\log 2Q.$$

Thus it suffices to bound

$$\sum_{s\leqslant Q}\frac{1}{s}x^{-1}\sum_{\substack{n\leqslant x\\ s|n}}|a_n|\sum_{\substack{t\leqslant Q/s\\ t|n/s}}1.$$

By Cauchy's inequality and (1.5), the sum over n is

$$\ll\biggl(x^{-1}\sum_{\substack{n\leqslant x\\ s|n}}\biggl(\sum_{\substack{t\leqslant Q/s\\ t|n/s}}1\biggr)^2\biggr)^{1/2}$$
$$\ll s^{-1/2}(\log 2Q)^{3/2},$$

and (2.1) follows. ■

Lemma 2.2. *Suppose that (1.4) holds. Then*

$$f(r,r)=\sum_{\substack{a=1\\ r|a}}^{q}f(q,(q,a)) \tag{2.3}$$

and

$$\sum_{r|q}\mu(r)f(sr,sr)=\phi(q)f(sq,s). \tag{2.4}$$

Proof. By (1.4)

$$f(r,r)=\lim_{x\to\infty}x^{-1}\sum_{\substack{n\leqslant x\\ r|n}}a_n.$$

The sum here is

$$\sum_{\substack{a=1\\ r|a}}^{q}\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}}a_n,$$

which gives (2.3). Then, on replacing r by sr, we obtain

$$\sum_{r|q}\mu(r)f(sr,sr)=\sum_{r|q}\mu(r)\sum_{\substack{a=1\\ sr|a}}^{sq}f(sq,(sq,a))$$
$$=\sum_{\substack{b=1\\ (q,b)=1}}^{q}f(sq,s).$$

■

Let

$$J(\beta)=\sum_{n\leqslant x}e(n\beta), \tag{2.5}$$

$$\nu(q)=\sum_{s|q}f(q,s)\mu(q/s) \tag{2.6}$$

and

$$\Delta(q,a,\beta) = G(a/q+\beta) - \nu(q)J(\beta). \tag{2.7}$$

Lemma 2.3. *Suppose that (1.4) holds. Then*

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} \Delta(q,a,\beta) \ll \sigma(q)\frac{x}{\Psi(x)}(1+x|\beta|), \tag{2.8}$$

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} |\Delta(q,a,\beta)|^2 \ll \frac{q^2x^2}{\Psi(x)^2}(1+x|\beta|)^2, \tag{2.9}$$

and when $(a,q)=1$,

$$\lim_{x\to\infty} x^{-1}G(a/q) = \nu(q). \tag{2.10}$$

Proof. We have

$$G(a/q+\beta) = \sum_{r=1}^{q} e(ar/q) \sum_{\substack{n\leqslant x\\ n\equiv r\ (\mathrm{mod}\ q)}} a_n e(n\beta)$$

and

$$\sum_{r=1}^{q} e(ar/q)f(q,(q,r)) = \sum_{s|q}\sum_{\substack{t=1\\(q/s,t)=1}}^{q/s} e\left(\frac{at}{q/s}f(q,s)\right)$$
$$= \sum_{s|q}\mu(q/s)f(q,s).$$

Therefore, by (2.5)–(2.7),

$$\Delta(q,a,\beta) = \sum_{r=1}^{q} e(ar/q)\Bigg(\sum_{\substack{n\leqslant x\\ n\equiv r\ (\mathrm{mod}\ q)}} a_n e(n\beta) - \sum_{n\leqslant x} f(q,(q,r))e(n\beta)\Bigg). \tag{2.11}$$

We concentrate first on (2.8). We sum the above over a with $1 \leqslant a \leqslant q$ and $(q,a)=1$. We take the sum over a inside the sum over r. The sum of $e(ar/q)$ over these a is $\sum_{s|(q,r)} s\mu(q/s)$. Then we interchange the sums over r and s and (2.11) becomes

$$\sum_{s|q} s\mu(q/s) \sum_{\substack{r=1\\ s|r}}^{q}\Bigg(\sum_{\substack{n\leqslant x\\ n\equiv r\ (\mathrm{mod}\ q)}} a_n e(n\beta) - \sum_{n\leqslant x} f(q,(q,r))e(n\beta)\Bigg).$$

By (2.3) this is

$$\sum_{s|q} s\mu(q/s) \sum_{\substack{n\leqslant x\\ s|n}} (a_n - f(s,s))e(n\beta),$$

and by (1.4) and partial summation this is

$$\ll \sigma(q)\frac{x}{\Psi(x)}(1+x|\beta|),$$

which gives the first part of the lemma.

We now transfer our attention to (2.9). By (2.11) and the orthogonality of the additive characters we have

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} |\Delta(q,a,\beta)|^2 \leqslant q\sum_{r=1}^{q}\Biggl(\sum_{\substack{n\leqslant x\\ n\equiv r\ (\mathrm{mod}\ q)}} a_n e(n\beta) - \sum_{n\leqslant x} f(q,(q,r))e(n\beta)\Biggr)^2.$$

We now apply partial summation to the sums over n and invoke (1.4) once more. This gives (2.9). To complete the proof of the lemma we simply observe that (2.9) implies that $\lim_{x\to\infty} x^{-1}\Delta(q,a,0)=0$. ∎

Lemma 2.4. *Suppose that (1.4) holds. Then*

$$q\sum_{r|q}\phi(q/r)f(q,r)^2 = \sum_{r|q}\phi(r)\Biggl(\sum_{s|r} f(r,s)\mu(r/s)\Biggr)^2.$$

Proof. We evaluate

$$\lambda = \lim_{x\to\infty} x^{-2}\sum_{a=1}^{q}\Biggl|\sum_{n\leqslant x} a_n e(an/q)\Biggr|^2$$

in two different ways. By the orthogonality of the additive characters, the sum over a is

$$q\sum_{m\leqslant x}\sum_{\substack{n\leqslant x\\ n\equiv m\ (\mathrm{mod}\ q)}} a_m a_n = q\sum_{a=1}^{q}\Biggl(\sum_{\substack{m\leqslant x\\ m\equiv a\ (\mathrm{mod}\ q)}} a_m\Biggr)^2,$$

and so

$$\lambda = q\sum_{a=1}^{q} f(a,(q,a))^2 = q\sum_{r|q}\phi(q/r)f(q,r)^2.$$

On the other hand, the sum over a is also

$$\sum_{r|q}\sum_{\substack{b=1\\(b,r)=1}}^{r}\Biggl|\sum_{n\leqslant x} a_n e(bn/r)\Biggr|^2,$$

and by (2.9) $\lim_{x\to\infty} x^{-1}\Delta(q,a,0)=0$. Thus, by (2.5)–(2.7),

$$\lambda = \sum_{r|q}\sum_{\substack{b=1\\(b,r)=1}}^{r}\Biggl(\sum_{s|r} f(r,s)\mu(r/s)\Biggr)^2,$$

which completes the proof. ∎

Lemma 2.5. *Assume (1.4) and (1.5), and that g is defined by (1.6). Then the series*

$$\sum_{q=1}^{\infty} g(q)$$

converges.

Proof. By the large sieve inequality (see, for example, Davenport 1980, § 27, theorem 2),

$$\sum_{q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} |G(a/q)|^2 \leqslant (x+Q^2)\sum_{n\leqslant x} a_n^2.$$

We divide both sides by x^2 and consider the limit superior as $x \to \infty$. Hence, by (1.5), (2.6), (2.10) and (1.6),

$$\sum_{q\leqslant Q} g(q) = \sum_{q\leqslant Q} \phi(q)\biggl(\sum_{s|q} f(q,s)\mu(q/s)\biggr)^2 \ll 1,$$

and this holds uniformly for all Q. ∎

3. Initial arrangements

We have

$$V(x,Q) = 2S_1 - 2S_2 + S_3 + [Q]\sum_{n\leqslant x} a_n^2, \tag{3.1}$$

where

$$S_1 = \sum_{q\leqslant Q}\sum_{n\leqslant x}\sum_{\substack{m<n\\q|n-m}} a_m a_n, \tag{3.2}$$

$$S_2 = \sum_{q\leqslant Q}\sum_{a=1}^{q} xf(q,(q,a)) \sum_{\substack{n\leqslant x\\n\equiv a\,(\mathrm{mod}\,q)}} a_n, \tag{3.3}$$

$$S_3 = \sum_{q\leqslant Q}\sum_{r|q} x^2\phi(q/r)f(q,r)^2. \tag{3.4}$$

We have

$$S_2 = \sum_{q\leqslant Q}\sum_{r|q} xf(q,r) \sum_{\substack{a=1\\(a,q)=r}}^{q} \sum_{\substack{n\leqslant x\\n\equiv a\,(\mathrm{mod}\,q)}} a_n$$

and the innermost double sum is

$$\sum_{\substack{m\leqslant x/r\\(m,q/r)=1}} a_{mr} = \sum_{s|q/r}\mu(s)\sum_{n\leqslant x/rs} a_{nrs}.$$

Hence, by (1.4),

$$S_2 = \sum_{r|q} f(q,r) \sum_{s|q/r} \mu(s) x^2 f(rs,rs) + O\biggl(\sum_{q\leqslant Q}\sum_{r|q} |f(q,r)| d(q/r) x^2/\Psi(x)\biggr).$$

Therefore, by lemmata 2.1 and 2.4

$$S_2 = \sum_{q\leqslant Q} x^2 \sum_{r|q} \phi(q/r) f(q,r)^2 + O(x^2\Psi(x)^{-1}(\log 2Q)^{7/2}).$$

Therefore, by (3.1) and (3.4)

$$V(x,Q) = 2S_1 - S_3 + [Q]\sum_{n\leqslant x} a_n^2 + O(x^2\Psi(x)^{-1}(\log 2Q)^{7/2}). \tag{3.5}$$

As is usual in these questions, the main part of our argument is concerned with the sum S_1.

For future reference observe that it may be supposed that x is sufficiently large.

4. The Farey dissection

Here we give only the conclusions as we follow exactly § 4 of Vaughan (this volume), to which we refer the interested reader for details. Let

$$F_q(\alpha) = \sum_{\substack{l\leqslant\sqrt{x}\\ q|l}} \biggl(\sum_{m\leqslant x/l} + \sum_{\sqrt{x}<m\leqslant\min(Q,x/l)}\biggr) e(\alpha l m). \tag{4.1}$$

Then

$$F_q(\alpha) \ll \frac{x\log(2\sqrt{x}/q)}{q+qx|\beta|} \quad (q\leqslant\sqrt{x}, |\beta|\leqslant \tfrac{1}{2}q^{-1}x^{-1/2}). \tag{4.2}$$

We suppose that R satisfies

$$2\sqrt{x}\leqslant R\leqslant \tfrac{1}{2}x \tag{4.3}$$

and define the major arc $\mathfrak{N}(q,a)$ by

$$\mathfrak{N}(q,a) = \biggl[\frac{a}{q} - q^{-1}(2R)^{-1}, \frac{a}{q} + q^{-1}(2R)^{-1}\biggr] \tag{4.4}$$

and let G be as in (1.10). Then

$$S_1 = S_4 + O(xR\log x), \tag{4.5}$$

where

$$S_4 = \sum_{q\leqslant x/R} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{N}(q,a)} F_q(\alpha)|G(\alpha)|^2\,\mathrm{d}\alpha. \tag{4.6}$$

5. The major arcs

Suppose that $1\leqslant a\leqslant q\leqslant x/R$, $(q,a)=1$ and $\alpha\in\mathfrak{N}(q,a)$. Then, by (4.4), $\alpha = a/q+\beta$, where $\beta\in I(q)$ with

$$I(q) = [-\tfrac{1}{2}q^{-1}R^{-1}, \tfrac{1}{2}q^{-1}R^{-1}]. \tag{5.1}$$

Hence, by (4.1) and (4.4)

$$\sum_{\substack{a=1\\(q,a)=1}}^{q} \int_{\mathfrak{N}(q,a)} F_q(\alpha)|G(\alpha)|^2 = \int_{I(q)} F_q(\beta) \sum_{\substack{a=1\\(q,a)=1}}^{q} |G(a/q+\beta)|^2 \, \mathrm{d}\beta.$$

By (2.6) and (2.7), the integrand here is

$$\nu(q)^2 F_q(\beta)|J(\beta)|^2 + \Delta_1(\beta) + \Delta_2(\beta),$$

where

$$\Delta_1(\beta) = \nu(q)F_q(\beta) 2\Re J(-\beta) \sum_{\substack{a=1\\(a,q)=1}}^{q} \Delta(q,a,\beta)$$

and

$$\Delta_2(\beta) = F_q(\beta) \sum_{\substack{a=1\\(a,q)=1}}^{q} |\Delta(q,a,\beta)|^2.$$

Hence, by (2.5), (2.6), (2.8), (2.9) and (4.2),

$$\Delta_1(\beta) \ll \sum_{r|q} |f(q,r)| \frac{\sigma(q)x^3 \log x}{q\Psi(x)} (1+x|\beta|)^{-1}$$

and

$$\Delta_2(\beta) \ll \frac{qx^3 \log x}{\Psi(x)^2}(1+x|\beta|).$$

Therefore, by (1.6) and (5.1),

$$\sum_{\substack{a=1\\(q,a)=1}}^{q} \int_{\mathfrak{N}(q,a)} F_q(\alpha)|G(\alpha)|^2 = \int_{I(q)} g(q)F_q(\beta)|J(\beta)|^2 + O\biggl(\sum_{r|q} |f(q,r)| \frac{\sigma(q)x^2(\log x)^2}{q\Psi(x)} + \frac{x^3 \log x}{\Psi(x)^2 R} + \frac{x^4 \log x}{q\Psi(x)^2 R^2}\biggr).$$

Hence, by (2.1),

$$S_4 = S_5 + O(x^2(\log x)^{9/2}\Psi(x)^{-1} + x^4(\log x)^2 R^{-2}\Psi(x)^{-2}), \tag{5.2}$$

where

$$S_5 = \sum_{q \leqslant x/R} \phi(q)g(q) \int_{I(q)} F_q(\beta)|J(\beta)|^2 \, \mathrm{d}\beta. \tag{5.3}$$

We now wish to replace each $I(q)$ by a unit interval. We cannot use (4.2) throughout the new range for β. However, by (4.1) we have the cruder estimate

$$F_q(\beta) \ll xq^{-1} \log x. \tag{5.4}$$

Then the error introduced by replacing $I(q)$ by a unit interval is

$$\ll \sum_{q \leqslant x/R} g(q) x R \log x \ll x R \log x.$$

We may also use (5.4) to estimate the contribution when we add the q in the range $x/R < q \leqslant \sqrt{x}$. Thus, by lemma 2.5 and (5.2),

$$S_5 = S_6 + O(xR\log x)$$

where

$$S_6 = \sum_{q\leqslant\sqrt{x}} \phi(q)g(q)\int_{-1/2}^{1/2} F_q(\beta)|J(\beta)|^2\,\mathrm{d}\beta. \tag{5.5}$$

Thus, by (4.5) and (5.2) S_1 differs from S_6 by an amount which is

$$\ll Rx\log x + x^2(\log x)^{9/2}\Psi(x)^{-1} + x^4(\log x)^2R^{-2}\Psi(x)^{-2}.$$

The optimal choice for R here is

$$R = \max(2\sqrt{x}, x(\log x)^{1/3}\Psi(x)^{-2/3})$$

and we observe that this is consonant with (4.3) since $\Psi(x) > \log x$. Hence

$$S_1 = S_6 + O(x^{3/2}\log x + x^2(\log x)^{9/2}\Psi(x)^{-1} + x^2(\log x)^{4/3}\Psi(x)^{-2/3}). \tag{5.6}$$

6. Completion of the proof of theorem 1.1

By (5.5), (4.1) and (1.10),

$$S_6 = \sum_{l\leqslant\sqrt{x}} h(l)\Bigg(\sum_{m\leqslant x/l} + \sum_{\sqrt{x}<m\leqslant\min(Q,x/l)}\Bigg)([x]-lm), \tag{6.1}$$

where

$$h(l) = \sum_{q|l} g(q). \tag{6.2}$$

By lemma 2.5,

$$\sum_{l\leqslant\sqrt{x}}\frac{h(l)}{l} = \sum_{r\leqslant\sqrt{x}}\frac{g(r)}{r}\sum_{m\leqslant x/r}\frac{1}{m} \ll \log x,$$

and similarly

$$\sum_{l\leqslant\sqrt{x}} h(l) \ll \sqrt{x}. \tag{6.3}$$

Thus the $[x]$ in (6.1) can be replaced by x with a total error $\ll x\log x$. A straightforward calculation shows that

$$\Bigg(\sum_{m\leqslant x/l} + \sum_{\sqrt{x}<m\leqslant\min(Q,x/l)}\Bigg)(x-lm)$$

is

$$\frac{x^2}{2l} + \frac{x}{2l}(\sqrt{x}-l)^2 - \frac{Q^2}{2l}\left(\frac{x}{Q}-l\right)^2 + O(x)$$

when $l \leqslant x/Q$ and is

$$\frac{x^2}{2l} + \frac{x}{2l}(\sqrt{x}-l)^2 + O(x)$$

when $x/Q < l \leqslant \sqrt{x}$. Hence, by (6.2),

$$2S_6 = x^2 \sum_{l \leqslant \sqrt{x}} \frac{h(l)}{l} + xW(\sqrt{x}) - Q^2 W(x/Q) + O(x^{3/2}), \tag{6.4}$$

where

$$W(X) = \sum_{l \leqslant X} \frac{h(l)}{l}(X-l)^2. \tag{6.5}$$

Therefore, by (3.5), (3.4), lemma 2.4, (1.6) and (5.6),

$$\begin{aligned} V(x,Q) = &-x^2 \sum_{\sqrt{x} < l \leqslant x/Q} \frac{h(l)}{l} + Q \sum_{n \leqslant x} a_n^2 + xW(\sqrt{x}) - Q^2 W(x/Q) \\ &+ O(x^{3/2}(\log x) + x^2(\log x)^{9/2}\Psi(x)^{-1} + x^2(\log x)^{4/3}\Psi(x)^{-2/3}). \end{aligned} \tag{6.6}$$

Let

$$C_0 = \sum_{q=1}^{\infty} \frac{g(q)}{q}, \tag{6.7}$$

and

$$C_1 = \gamma C_0 - \sum_{q=1}^{\infty} \frac{g(q)\log q}{q}, \tag{6.8}$$

where γ is Euler's constant. By (6.2),

$$\sum_{l \leqslant X} \frac{h(l)}{l} = \sum_{q \leqslant X} \frac{g(q)}{q} \sum_{m \leqslant X/q} \frac{1}{m} = \sum_{q \leqslant X} \frac{g(q)}{q}\left(\log\frac{X}{q} + \gamma + O(q/X)\right).$$

Hence, by lemma 2.5, (6.6) and (6.7),

$$\sum_{l \leqslant X} \frac{h(l)}{l} = C_0 \log X + C_1 + O(X^{-1}\log X).$$

Therefore, by (6.6),

$$\begin{aligned} V(x,Q) = Q &\sum_{n \leqslant x} a_n^2 - C_0 x^2 \log\frac{Q}{\sqrt{x}} + xW(\sqrt{x}) - Q^2 W(x/Q) \\ &+ O(x^{3/2}\log x + x^2(\log x)^{9/2}\Psi(x)^{-1} + x^2(\log x)^{4/3}\Psi(x)^{-2/3}). \end{aligned} \tag{6.9}$$

The final stage of the proof of theorem 1.1 is the investigation of $W(X)$. We have

$$\sum_{m \leqslant X} \frac{1}{m}(X-m)^2 = \int_1^X \left(\frac{X^2}{u^2} - 1\right)[u]\,\mathrm{d}u.$$

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Let $B_1(u) = u - [u] - \frac{1}{2}$ and $B_2(u) = \frac{1}{2}(u-[u])^2 - \frac{1}{2}(u-[u]) + \frac{1}{12}$. Then, by repeated partial summation the above becomes

$$\sum_{m \leqslant X} \frac{1}{m}(x-m)^2 = X^2 \log X - X^2 + X - \int_1^X \left(\frac{X^2}{u^2} - 1\right) B_1(u)\,\mathrm{d}u$$
$$= X^2(\log X - 1) + X + (X^2-1)B_2(1) - \int_1^X \frac{2X^2}{u^3} B_2(u)\,\mathrm{d}u. \tag{6.10}$$

Hence

$$\sum_{m \leqslant X} \frac{1}{m}(X-m)^2 = X^2 \log X + C_2 X^2 + X + O(1), \tag{6.11}$$

where

$$C_2 = -\frac{11}{12} - 2\int_1^\infty \frac{B_2(u)}{u^3}\,\mathrm{d}u. \tag{6.12}$$

Therefore, by (6.2) and (6.5),

$$W(Y) = \sum_{q \leqslant Y} q g(q) \sum_{r \leqslant Y/q} \left(\frac{Y}{q} - r\right)^2$$
$$= Y^2 \sum_{q \leqslant Y} (\log(Y/q) + C_2)\frac{g(q)}{q} + Y \sum_{q \leqslant Y} g(q) + O\biggl(\sum_{q \leqslant Y} q g(q)\biggr). \tag{6.13}$$

The error here is

$$\int_0^Y \sum_{u < q \leqslant Y} g(q)\,\mathrm{d}u \leqslant E(Y),$$

and completing each sum in (6.13) to infinity introduces an error

$$\ll Y^2 \sum_{q > Y} g(q) \frac{\log(q/Y) + 1}{q} + Y \sum_{q > Y} g(q).$$

The factor $(\log(q/Y) + 1)/q$ is a decreasing function of q when $q \geqslant Y$. Thus the introduced error is

$$\ll Y \sum_{q > Y} g(q) \leqslant E(Y).$$

Hence

$$W(Y) = Y^2(\log Y + C_2)C_0 - Y^2 \sum_{q=1}^\infty \frac{g(q)\log q}{q} + Y \sum_{q=1}^\infty g(q) + O(E(Y)).$$

Theorem 1.1 now follows from this, (6.9) and lemma 2.5.

7. The proof of theorem 1.2

By (6.10)

$$\sum_{m \leqslant X} \frac{1}{m}(X-m)^2 = X^2 \log X + C_2 X^2 + X + C_3 + \int_X^\infty \frac{2X^2}{u^3} B_2(u)\,\mathrm{d}u,$$

where C_2 is given by (6.12) and

$$C_3 = -\tfrac{1}{12}.$$

Therefore

$$W(Y) = N(Y) - P_1(Y) + P_2(Y) + P_3(Y),$$

where

$$N(Y) = \sum_{r=1}^{\infty} g(r)\left(\frac{Y^2}{r}\log\frac{Y}{r} + C_2\frac{Y^2}{r} + Y\right), \tag{7.1}$$

$$P_1(Y) = \sum_{r>Y} g(r)\left(\frac{Y^2}{r}\log\frac{Y}{r} + C_2\frac{Y^2}{r} + Y\right),$$

$$P_2(Y) = C_3 \sum_{r\leqslant Y} rg(r)$$

and

$$P_3(Y) = \int_1^{\infty} \frac{2Y^2}{u^3} B_2(u) \sum_{Y/u<r\leqslant Y} \frac{g(r)}{r}\,\mathrm{d}u.$$

By (1.12) and partial summation

$$\sum_{r>y} \frac{g(r)}{r} \sim \frac{c\eta}{1+\eta} y^{-1-\eta},$$

$$\sum_{r>y} \frac{g(r)}{r}\log\frac{r}{y} \sim \frac{c\eta}{(1+\eta)^2} y^{-1-\eta}$$

and

$$\sum_{r\leqslant y} rg(r) \sim \frac{c\eta}{1-\eta} y^{1-\eta}$$

as $y \to \infty$. Thus

$$W(Y) = N(Y) + P_4(Y) + P_5(Y),$$

where

$$P_4(Y) = cC_4 Y^{1-\eta},$$

with

$$C_4 = \frac{2\eta}{1+\eta}\int_1^{\infty} \frac{B_2(u)}{u^{2-\eta}}\,\mathrm{d}u - \frac{\eta}{(1+\eta)^2} - \frac{11\eta}{12+12\eta} - 1 - \frac{\eta}{12-12\eta},$$

and

$$P_5(Y) = o(Y^{1-\eta})$$

as $Y \to \infty$. To obtain the constant C_4 in a more convenient form we observe that C_4 depends only on η and that the above analysis holds in the special case $g(r) = \eta r^{-1-\eta}$, and then $c = 1$. On the other hand, in this case we have

$$W(Y) = \frac{1}{2\pi \mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \eta\zeta(s+1)\zeta(s+2+\eta)\frac{Y^{2+s}}{s(s+1)(s+2)}\,\mathrm{d}s$$

and it follows from standard estimates for the Riemann zeta function that the path of integration can be moved to the left of the line $\Re s = -1-\eta$. In doing so one picks up contributions from the poles at $s=0$, $s=-1$ and $s=-1-\eta$. The residues from the poles at $s=0$ and $s=-1$ give precisely $N(Y)$ and the residue at $-1-\eta$ is

$$\frac{Y^{1-\eta}2\zeta(-\eta)}{1-\eta^2}.$$

Moreover, the contribution from the new path of integration is

$$o(Y^{1-\eta}).$$

Hence

$$C_4 = \frac{2\zeta(-\eta)}{1-\eta^2}.$$

Thus

$$W(Y) = N(Y) + c\frac{2\zeta(-\eta)}{1-\eta^2}Y^{1-\eta} + o(Y^{1-\eta}).$$

Armed with this new estimate we return to $V(x,Q)$. By (6.9), (7.1) and (6.7),

$$V(x,Q) = Q\sum_{n\leqslant x} a_n^2 - Qx\sum_{q=1}^{\infty} g(q) + Q^2 M(x/Q)$$
$$+ O(x^{3/2}\log x + x^2(\log x)^{9/2}\Psi(x)^{-1} + x^2(\log x)^{4/3}\Psi(x)^{-2/3}),$$

where

$$M(y) \sim c\frac{-2\zeta(-\eta)}{1-\eta^2}y^{1-\eta}$$

as $y\to\infty$.

It is easily verified that

$$Qx^{(2+\eta)/(2+2\eta)} \leqslant x^{3/2} + Q^{1+\eta}x^{1-\eta}$$

and theorem 1.2 follows at once.

8. The proof of theorem 1.3

It suffices to estimate $A(x;q,a)$ accurately. Clearly $A(x;q,a) = B(x;q,a) + O(1)$, where $B(x;q,a)$ is the number of m with $m\leqslant x/q$ and $\{\lambda(mq+a)\} < \theta$. Choose b and r so that $|\lambda r - b| \leqslant 1/r$, $(b,r) = 1$ and $r \asymp x^{2/3}$. Let $r_1 = r/(q,r)$ and $q_1 = q/(q,r)$. For each j with $0\leqslant j < r_1$ the number of m with $m\leqslant x/q$ and $[\lambda a r_1] + bmq_1 \equiv j \pmod{r_1}$ is

$$\frac{x}{qr_1} + O(1).$$

Moreover, for such an m

$$\{\lambda(mq+a)\} = \{j/r_1 + mq(\lambda - b/r) + \{\lambda a r_1\}/r_1\},$$

and with the exception of $\ll x/r+1$ values of j this is

$$j/r_1 + \eta(x/r^2 + 1/r),$$

where $|\eta|\leqslant 1$. Thus

$$B(x;q,a) = \sum_{0\leqslant j\leqslant \theta r_1}\left(\frac{x}{qr_1} + O(1)\right) + O\left(\left(\frac{x}{r}+1\right)\left(\frac{x}{qr_1}+1\right)\right),$$

and the desired conclusion follows.

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